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Axially symmetric motion of a stratified, rotating fluid in a spherical annulus of narrow gap

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A linear theory is presented for the steady, axially symmetric motion of a stratified fluid in a narrow, rotating spherical annulus with a spherically symmetric gravitational field.

The fluid is driven by a combination of differential rotation of the two shells and differential heating applied at the surfaces of the spheres.

It is shown that the effect of stratification becomes increasingly important at lower latitudes with the Ekman layers on the spheres' surfaces fading in strength as the geostrophic interior velocities themselves tend toward the shell speeds at lower latitudes.

The singularities in the geostrophic solutions at the equator are removed by a boundary layer whose detailed structure depends on the ratio of horizontal to vertical mixing coefficients of momentum and heat.

1. Introduction

The fascinating nature of the dynamics of rapidly rotating fluids is due primarily to the dominance of Coriolis forces acting on the fluid's motion in planes perpendicular to the rotation axis. For a fluid constrained to flow in thin sheets on the surface of a sphere (a case of obvious geophysical interest) the geometrical constraint inhibits velocities and accelerations in the direction perpendicular to the sphere's surface, and so the important Coriolis forces are those which act on the velocities tangential to the sphere's surface. Since these horizontal Coriolis forces depend on the local component of the rotation perpendicular to the sphere's surface and hence on the sine of the latitude, the Coriolis forces must become small and finally vanish as the equator is approached.

The equatorial region is thus a very special region in which the dynamical nature of the flow although affected by rotation may be quite different from that at higher latitudes where the rotational constraint more powerfully asserts itself. Questions of obvious interest then are the extent of the equatorial region, the nature of its dynamics and the manner by which the motion at higher latitudes, dominated by rotation, merges into the equatorial region.

In an effort to throw some light on these questions this paper considers the steady, axially symmetric motion of a thin stably stratified layer of fluid between two concentric spherical shells. The fluid motion is produced by both differential rotation of the shells and an applied surface differential heating. The introduction of stratification is an attempt to consider a model of greater geophysical interest. The motion of a homogeneous fluid in a similar geometry has been considered by Stewartson (1966) (without any restriction of the thinness of the shell of fluid.) The motion, as in the study of Stewartson's, is limited to almost rigid rotation; that is, the motion relative to the mean rotation of the spheres is assumed to be sufficiently small so that a linear dynamical theory is valid.

The axially symmetric motion investigated in this paper may also be of interest in the context of tropical meteorology. Although the general circulation of the earth's atmosphere in middle latitudes is very asymmetric, this asymmetry becomes less pronounced in equatorial regions where an axially symmetric model may be appropriate (Palmen 1963) as a first approximation. Moreover, as Charney (1968) has pointed out theories based on instability arguments for the development of smaller scale features of the tropical circulation (such as the inter-tropical convergence zone) which depend on other factors such as heat released by condensation of water vapour, may well require first the results of a 'dry' tropical circulation as a starting point for the mean field.

In a recent paper (Barcilon & Pedlosky 1967) it was shown that a critical parameter governing the nature of the bulk of the fluid motion for a stably stratified rotating fluid in 'flat' geometries was the ratio $\sigma S/E^{\frac{1}{2}}$, where

$$\sigma S = \frac{\nu\alpha g \Delta T}{\kappa \Omega^2 L} \quad \text{and} \quad E = \frac{\nu}{\Omega L^2},$$

where α is the coefficient of thermal expansion, g the acceleration of gravity, $\Delta T/L$ the basic stable temperature gradient, Ω the mean rotation, ν the kinematic viscosity, κ the thermal conductivity while L is a characteristic scale. When the ratio is large the fluid acts, in the main, as if it were substantially stratified. Now $\sigma S/E^{\frac{1}{2}}$ is proportional to $\Omega^{-\frac{3}{2}}$, and one might suppose that for the sphere Ω should be replaced by $\Omega \sin\theta$ where θ is latitude. This is indeed what will be shown to be the case by detailed calculation below, with the result that the fluid acts more and more substantially stratified in the sense discussed in Barcilon & Pedlosky (1967) as the equator is approached.

2. The model

Consider a spherical annulus of thickness D, the inner sphere of which has radius R, where $D \ll R$. The spherical shells are very nearly rotating at angular velocity Ω about an axis passing through the poles of the sphere. The co-ordinate system is chosen so that the directions north, east, etc., have their usual meanings. The co-ordinate θ measures latitude, r is the spherical radius of a field point while (u, v, w) are the velocities to the east, north and vertical, respectively. The fluid in the spherical annulus is assumed to be stably stratified. The direction of the effective gravitational force is taken to be radial which implies that the centrifugal force due to the mean rotation is neglected. This requires that $\Omega^2R^2/gD \ll 1$. Furthermore, the fluid density variations are assumed to be linearly related to its temperature variations which in turn are considered small enough for a Boussinesq approximation to hold, i.e. for the fluid to be very nearly incompressible, and only the buoyancy effects of variable density are considered in the momentum equations. In order to consider a model of geophysical relevance the viscosity and diffusivity of temperature must be considered as parametrizations of small-scale turbulent mixing processes. It is natural therefore to allow that the transport coefficients need not be the same for horizontal as well as vertical fluxes of momentum and heat. The theory will be developed with the ratio of the horizontal to vertical mixing coefficients as a free parameter and the solutions in various parameter régimes will be discussed in detail. It should be pointed out that the difficulty in producing the spherically symmetric gravitational force of this theory in the laboratory de-emphasizes the importance of what might be called the laboratory situation in so far as the viscosity coefficients are concerned.

Non-dimensional variables, denoted by primes are defined as follows: if p, ρ and T are the pressure, density and temperature of the fluid then

$$\begin{aligned} r &= R+\delta R z', \\ (u,v) &= U(u'(\theta,z'),\, v'(\theta,z')), \\ w &= \delta U w'(\theta,z'), \quad T = T_0+\Delta T_v z' + (2\Omega R U/g\alpha D)\, T'(\theta,z'), \\ \rho &= \rho_0(1-\alpha(T-T_0)), \\ p &= p_0-\rho_0 g D z' + \alpha \Delta T_v \rho_0 g D^2 \tfrac{1}{2} z'^2 + 2\Omega U R \rho_0 p'(\theta,z'), \end{aligned}$$

where $\delta \equiv D/R$, α is the coefficient of thermal expansion, U is a horizontal velocity scale which will later be related to the boundary conditions and T_0, ρ_0 and p_0 are constant reference levels for T, ρ and p. Note that the dependent variables are functions of $z' = r-R/\delta R$.

The equations of motion in terms of the non-dimensional variables appropriate for linear, axially symmetric steady motion (after dropping the primes from the dimensionless co-ordinates) are

$$-v\sin\theta+\delta w\cos\theta = \frac{E_H}{2}\left[\frac{1}{\cos\theta}\frac{\partial}{\partial\theta}\cos\theta\frac{\partial u}{\partial\theta}-\frac{u}{\cos^2\theta}\right]+\frac{E_V}{2}\left[\frac{\partial^2 u}{\partial z^2}+2\delta\frac{\partial u}{\partial z}\right], \tag{2.1a}$$

$$u\sin\theta = -\frac{\partial p}{\partial\theta}+\frac{E_H}{2}\left[\frac{1}{\cos\theta}\frac{\partial}{\partial\theta}\cos\theta\frac{\partial v}{\partial\theta}-\frac{v}{\cos^2\theta}\right]+\frac{E_V}{2}\left[\frac{\partial^2 v}{\partial z^2}+2\delta\frac{\partial v}{\partial z}\right], \tag{2.1b}$$

$$\begin{aligned} -\delta u\cos\theta = -\frac{\partial p}{\partial z}+T+\frac{\delta^2 E_H}{2}\left[\frac{1}{\cos\theta}\frac{\partial}{\partial\theta}\cos\theta\frac{\partial w}{\partial\theta}-\frac{w}{\cos^2\theta}\right] \\ +\delta^2\frac{E_V}{2}\left[\frac{\partial^2 w}{\partial z^2}+2\delta\frac{\partial w}{\partial z}\right], \end{aligned} \tag{2.1c}$$

$$wS = \frac{E_H}{2\sigma_H}\frac{1}{\cos\theta}\frac{\partial}{\partial\theta}\cos\theta\frac{\partial T}{\partial\theta}+\frac{E_V}{2\sigma_V}\left[\frac{\partial^2 T}{\partial z^2}+2\delta\frac{\partial T}{\partial z}\right], \tag{2.1d}$$

$$\frac{\partial w}{\partial z}+2\delta w+\frac{1}{\cos\theta}\frac{\partial}{\partial\theta}(\cos\theta v) = 0. \tag{2.1e}$$

In these equations the spherical radius r has been approximated by the constant R when undifferentiated.

The following non-dimensional parameters have emerged:

$$E_H = \nu_H/\Omega R^2, \quad \text{the 'horizontal' Ekman number;}$$

$$E_V = \nu_V/\Omega D^2, \quad \text{the 'vertical' Ekman number;}$$

$$\delta = D/R, \quad \text{the aspect ratio;}$$

$$S = \frac{\alpha g \Delta T_V D}{\Omega^2 R^2}, \quad \text{the stratification parameter;}$$

$$\sigma_H = \nu_H/\kappa_H, \quad \text{the 'horizontal' Prandtl number;}$$

$$\sigma_V = \nu_V/\kappa_V, \quad \text{the 'vertical' Prandtl number;}$$

where ν_V, κ_V; ν_H, κ_H are the mixing coefficients for momentum and heat in the vertical and horizontal directions respectively.

The boundary conditions are:

$$(u, v, w) = (u_T(\theta), 0, 0) \quad \text{on} \quad z = 1, \tag{2.2a}$$

$$(u, v, w) = (u_L(\theta), 0, 0) \quad \text{on} \quad z = 0, \tag{2.2b}$$

while on the outer and inner spheres the radial heat flux will be specified, i.e.

$$\frac{\partial T}{\partial z} = H_T(\theta) \quad \text{on} \quad z = 1, \tag{2.2c}$$

$$\frac{\partial T}{\partial z} = H_L(\theta) \quad \text{on} \quad z = 0. \tag{2.2d}$$

Naturally the solution must satisfy conditions of regularity at the poles.

In the analysis which follows, E_H, E_V, S and δ will all be taken to be small parameters, but for convenience the additional restriction that

$$\delta \ll E_V^{\frac{1}{2}}$$

will be observed.

The boundary condition (2.2a) could easily be replaced by a condition specifying the stress on $z = 1$, which might be more relevant in a geophysical context. The condition (2.2a) is chosen because the resulting motions provide a more interesting interplay between the convection of heat and the dynamics. The alternate possibility of specifying the stress on $z = 1$, while leading to a rather trivial interior flow, in fact will yield the same sort of equatorial problem as given by the conditions (2.2). This will be discussed below.

Since E_V and E_H are small the approximate solutions to be found will be of boundary-layer type.

Finally, the scaling velocity U is chosen such that

$$U = \frac{\alpha g D^2}{2\Omega R} H_0,$$

where H_0 is a characteristic value of the latitudinally varying applied heat flux.

3. The Ekman layers

In regions where $\theta = O(1)$, i.e. removed from the equator, the dynamic variables may be written as

$$u = u_I(\theta, z) + \tilde{u}_T(\theta, \zeta_1) + \tilde{u}_L(\theta, \zeta_2), \tag{3.1a}$$

$$v = E_V v_I(\theta, z) + \tilde{v}_T(\theta, \zeta_1) + \tilde{v}_L(\theta, \zeta_2), \tag{3.1b}$$

$$w = E_V^{\frac{1}{2}} w_I(\theta, z) + E_V^{\frac{1}{2}} \tilde{w}_T(\theta, \zeta_1) + E_V^{\frac{1}{2}} \tilde{w}_L(\theta, \zeta_2), \tag{3.1c}$$

$$p = p_I(\theta, z) + \delta E_V^{\frac{1}{2}} \tilde{p}_T(\theta, \zeta_1) + \delta E_V^{\frac{1}{2}} \tilde{p}_L(\theta, \zeta_2), \tag{3.1d}$$

$$T = T_I(\theta, z) + E_V^{\frac{1}{2}} \sigma_V S \tilde{T}_T(\theta, \zeta_1) + E_V^{\frac{1}{2}} \sigma_V S \tilde{T}_L(\theta, \zeta_2). \tag{3.1e}$$

The subscripted I variables represent the dynamic variables in the interior of the fluid removed from the direct viscous effects of the boundaries. The tilde variables represent the necessary correction functions which must be added to the interior variables within the Ekman layers near $z = 0$ and $z = 1$. The T subscripted tilde variables are the boundary layer corrections required near $z = 1$ and go to zero as

$$\zeta_1 = (1 - z)/E_V^{\frac{1}{2}}$$

becomes large. Similarly the L subscripted tilde variables represent the Ekman layer corrections near $z = 0$ and go to zero as

$$\zeta_2 = z/E_V^{\frac{1}{2}}$$

becomes large. The characteristic thickness of the Ekman layer is, of course $E_V^{\frac{1}{2}}$.

The equations for the lowest-order correction functions in the layer near $z = 0$, for example, are,

$$-\tilde{v}_L \sin\theta = \frac{1}{2} \frac{\partial^2 \tilde{u}_L}{\partial \zeta_2^2}, \tag{3.2a}$$

$$\tilde{u}_L \sin\theta = \frac{1}{2} \frac{\partial^2 \tilde{v}_L}{\partial \zeta_2^2}, \tag{3.2b}$$

$$\delta \frac{\partial \tilde{p}_L}{\partial \zeta_2} = \delta \tilde{u}_L \cos\theta + \sigma_V S E_V \tilde{T}_L, \tag{3.2c}$$

$$\tilde{w}_L = \frac{1}{2} \frac{\partial^2 \tilde{T}_L}{\partial \zeta_2^2}, \tag{3.2d}$$

$$\frac{\partial \tilde{w}_L}{\partial \zeta_2} = -\frac{1}{\cos\theta} \frac{\partial}{\partial \theta} \cos\theta \tilde{v}_L, \tag{3.2e}$$

which, in conjunction with (2.2*b*), yield as solutions

$$\tilde{u}_L = (u_I(\theta, 0) - u_L)\, e^{-\zeta_2 |\sin\theta|^{\frac{1}{2}}} \cos(\zeta_2 |\sin\theta|^{\frac{1}{2}}), \tag{3.3a}$$

$$\tilde{v}_L = (u_I(\theta, 0) - u_L)\, e^{-\zeta_2 |\sin\theta|^{\frac{1}{2}}} \sin(\zeta_2 |\sin\theta|^{\frac{1}{2}}) \operatorname{sgn}\theta, \tag{3.3b}$$

$$\tilde{w}_L = \frac{\operatorname{sgn}\theta}{2^{\frac{1}{2}} \cos\theta} \frac{\partial}{\partial\theta} \left[(u_I(0, \theta) - u_L) \frac{\cos\theta}{|\sin\theta|^{\frac{1}{2}}} e^{-\zeta_2 |\sin\theta|^{\frac{1}{2}}} \sin(\zeta_2 |\sin\theta|^{\frac{1}{2}} + \tfrac{1}{4}\pi) \right], \tag{3.3c}$$

where

$$\operatorname{sgn}\theta \equiv \sin\theta / |\sin\theta|.$$

Since $w_I(\theta, 0) + \tilde{w}_L(\theta, 0) = 0$, we must have

$$w_I(\theta, 0) = -\frac{\operatorname{sgn}\theta}{2\cos\theta} \frac{\partial}{\partial\theta} \frac{\cos\theta (u_I(\theta, 0) - u_L)}{|\sin\theta|^{\frac{1}{2}}}, \tag{3.4}$$

which provides one boundary condition for the interior flow on $z = 0$. According to (3.1c) the correction to the heat flux in the Ekman layer is $O(\sigma_V S)$; thus if $\sigma_V S \ll 1$ the interior temperature must itself satisfy (2.2d), i.e.

$$(\partial T_I/\partial z)(\theta, 0) = H_L(\theta), \tag{3.5}$$

which provides the other boundary condition for the interior flow on $z = 0$. In fact the estimate for the amplitude for the Ekman layer correction temperature is conservative for, as we shall see, when $\sigma_V S \gg E_V^{\frac{1}{2}}$, w is $O(E_V/\sigma_V S)$ so that the heat flux correction due to the Ekman layer is never greater than $E_V^{\frac{1}{2}}$.

As the equator is approached the Ekman layer analysis becomes increasingly invalid. The thickness of the layer is proportional to $|\sin\theta|^{-\frac{1}{2}}$. In this linear analysis the approximations fail when

$$\theta = O(\delta^2 E_V)^{\frac{1}{5}},$$

which is when the Coriolis force $\delta w \cos\theta$ in (2.1a) can no longer be neglected. When $\delta = O(1)$ and $\nu_H = \nu_V$ this condition becomes identical with the criterion for the validity of the Ekman layer results deduced by Stewartson (1966).

We shall see that, because of the stratification, this limit in latitude on the Ekman layer validity is too conservative. One of the more interesting features of the following analysis (§ 4) is the effect that stratification has in extending the region of validity of the Ekman layer results.

The boundary layer near $z = 1$ is essentially the same as near $z = 0$ and yields as boundary conditions for the interior flow

$$w_I(\theta, 1) = \frac{\operatorname{sgn}\theta}{2\cos\theta}\frac{\partial}{\partial\theta}\frac{(u_I(\theta,1) - u_T)\cos\theta}{|\sin\theta|^{\frac{1}{2}}}, \tag{3.6}$$

$$(\partial T_I/\partial z)(\theta, 1) = H_T(\theta). \tag{3.7}$$

4. The geostrophic interior

In the interior of the fluid, at latitudes such that $\theta = O(1)$, the dynamic variables are represented by the I subscripted variable which satisfy the following interior equations to lowest order if $\delta \ll E_V^{\frac{1}{2}}$.

$$u_I \sin\theta = -\partial p_I/\partial\theta, \tag{4.1a}$$

$$-v_I \sin\theta = \frac{1}{2}\left[\frac{1}{\cos\theta}\frac{\partial}{\partial\theta}\cos\theta\frac{\partial u_I}{\partial\theta} - \frac{u_I}{\cos^2\theta} + \frac{E_H}{E_V}\frac{\partial^2 u_I}{\partial z^2}\right], \tag{4.1b}$$

$$T_I = \partial p_I/\partial z, \tag{4.1c}$$

$$\frac{\partial w_I}{\partial z} + \frac{E_V^{\frac{1}{2}}}{\cos\theta}\frac{\partial}{\partial\theta}\cos\theta\, v_I = 0, \tag{4.1d}$$

$$2w_I\frac{\sigma_V S}{E_V^{\frac{1}{2}}} = \frac{\partial^2 T_I}{\partial z^2} + \frac{E_H}{E_V}\frac{\sigma_V}{\sigma_H}\frac{1}{\cos\theta}\frac{\partial}{\partial\theta}\cos\theta\frac{\partial T_I}{\partial\theta}. \tag{4.1e}$$

Thus to $O(E_V^{\frac{1}{2}})$ w_I is a function only of θ and with (3.4) and (3.6) it may be written in terms of T_I, viz:

$$w_I(\theta) = -\frac{\operatorname{sgn}\theta}{4\cos\theta}\frac{\partial}{\partial\theta}\frac{\cos\theta}{|\sin\theta|^{\frac{1}{2}}}\left\{(u_T - u_L) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\int_0^1 T(\theta, z')\,dz'\right\}. \tag{4.2}$$

It is convenient to introduce the variable G, defined as follows,

$$G = \frac{\partial T_I}{\partial \theta} + \frac{\sigma_V S}{2\gamma^2 E_V^{\frac{1}{2}} |\sin\theta|^{\frac{3}{2}}} \int_0^1 \frac{\partial T_I}{\partial \theta}(\theta, z')\, dz', \tag{4.3}$$

where $\gamma^2 = E_H \sigma_V / E_V \sigma_H$. In terms of G, $\partial T_I/\partial\theta$ is

$$\frac{\partial T_I}{\partial \theta} = G - \frac{\sigma_V S}{2\gamma^2 E_V^{\frac{1}{2}} |\sin\theta|^{\frac{3}{2}}} (1 + \sigma_V S/2\gamma^2 E_V^{\frac{1}{2}} |\sin\theta|^{\frac{3}{2}})^{-1} \int_0^1 G(\theta, z')\, dz'. \tag{4.4}$$

The reason for introducing G is that it satisfies the relatively simple equation, derivable from (4.1c), (4.2) and (4.3), viz.

$$\frac{\partial^2 G}{\partial z^2} + \gamma^2 \left\{ \frac{1}{\cos\theta} \frac{\partial}{\partial \theta} \cos\theta \frac{\partial G}{\partial \theta} - \frac{G}{\cos^2\theta} \right\} = -\frac{\operatorname{sgn}\theta\, \sigma_V S}{2E_V^{\frac{1}{2}}} \frac{\partial}{\partial\theta} \frac{1}{\cos\theta} \frac{\partial}{\partial\theta} \frac{\cos\theta (u_T - u_L)}{|\sin\theta|^{\frac{1}{2}}}. \tag{4.5}$$

The solution for G may be found in a series of associated Legendre functions of order 1. The solution for the more physically meaningful interior temperature gradient can be found and can be shown to be

$$\begin{aligned} \frac{\partial T_I}{\partial\theta} = &-\frac{\lambda(\theta)}{1+\lambda(\theta)}(u_T - u_L)\sin\theta \\ &+ \sum_{n=1}^{\infty} \left[\alpha_n \left\{ \frac{\cosh l_n(z-\frac{1}{2})}{\cosh \frac{1}{2} l_n} - \frac{\lambda(\theta)}{1+\lambda(\theta)} \frac{2}{l_n} \tanh \tfrac{1}{2} l_n \right\} \right. \\ &\left. + \beta_n \frac{\sinh l_n (z-\frac{1}{2})}{\sinh \frac{1}{2} l_n} \right] P_n^{(1)}(\sin\theta), \end{aligned} \tag{4.6}$$

where

$$\lambda(\theta) = \sigma_V S / 2\gamma^2 |\sin\theta|^{\frac{3}{2}} E_V^{\frac{1}{2}},$$

$$l_n^2 = n(n+1)\gamma^2.$$

Since

$$u_I(\theta, 1) + u_I(\theta, 0) = u_L + u_T, \tag{4.7}$$

from (3.4), (3.6) and (4.1d) while

$$\sin\theta \frac{\partial u_I}{\partial z} = -\frac{\partial T_I}{\partial \theta} \tag{4.8}$$

from (4.1a) and (4.1c), $u_I(\theta, z)$ may be found from (4.6) and is

$$\begin{aligned} u_I = &\tfrac{1}{2}(u_T + u_L) + \frac{\lambda(\theta)}{1+\lambda(\theta)}(u_T - u_L)(z - \tfrac{1}{2}) \\ &- \sum_{n=1}^{\infty} \left[\frac{\alpha_n}{l_n} \left\{ \frac{\sinh l_n (z-\frac{1}{2})}{\cosh\frac{1}{2} l_n} - \frac{\lambda(\theta)}{1+\lambda(\theta)} \frac{2}{l_n} (z - \tfrac{1}{2}) \tanh \tfrac{1}{2} l_n \right\} \right. \\ &\left. + \frac{\beta_n}{l_n} \left\{ \frac{\cosh l_n (z-\frac{1}{2})}{\sinh \frac{1}{2} l_n} - \coth \tfrac{1}{2} l_n \right\} \right] \frac{P_n^{(1)}(\sin\theta)}{\sin\theta} \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} w_I = &-\frac{\operatorname{sgn}\theta}{4\cos\theta} \frac{\partial}{\partial\theta} \left[\frac{\cos\theta}{|\sin\theta|^{\frac{1}{2}}} \frac{u_T - u_L}{1+\lambda(\theta)} \right] \\ &- \frac{1}{4\cos\theta} \frac{\partial}{\partial\theta} \frac{\cos\theta}{|\sin\theta|^{\frac{3}{2}}} \sum_{n=1}^{\infty} \alpha_n \frac{2}{l_n} \frac{\tanh\frac{1}{2} l_n}{1+\lambda(\theta)} P_n^{(1)}(\sin\theta). \end{aligned} \tag{4.10}$$

The coefficients α_n and β_n can be found using (3.5) and (3.7), i.e.

$$\alpha_n = \frac{2n+1}{2}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{H_T(\theta)+H_L(\theta)}{2} P_n(\sin\theta)\cos\theta\, d\theta \frac{\coth\frac{1}{2}l_n}{l_n},$$

$$\beta_n = \frac{2n+1}{2}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{H_T(\theta)-H_L(\theta)}{2} P_n(\sin\theta)\cos\theta\, d\theta \frac{\tanh\frac{1}{2}l_n}{l_n}.$$

There are several features of interest in the general nature of this interior solution.

The most obvious feature is the singularity in u_I as θ tends to zero. This occurs for those terms in the sum in (4.9) for which $P_n^{(1)}(0) \neq 0$, i.e. for n odd. Since $P_n^{(1)}(\sin\theta)$ is an even function of θ for n odd that part of the solution for u_I which is singular at $\theta = 0$ is odd about the equator and is $O(\theta^{-1})$ as the equator is approached. It is also interesting to observe that this singularity is present only in that part of the solution which is thermally driven. If $H_L = H_T = 0$ then $\alpha_n = \beta_n = 0$ and the solution for u_I would consist solely of the first two terms of (4.9) which are *non-singular* across the equator. This mechanically driven solution u_{IM},

$$u_{IM} = \tfrac{1}{2}(u_T+u_L) + \frac{\lambda(\theta)}{1+\lambda(\theta)}(u_T-u_L)(z-\tfrac{1}{2}) \tag{4.11}$$

is quite simple and informative. It is remarkably similar to the solution found for a rotating stratified *cylinder* of fluid by Barcilon & Pedlosky (1967). When the fluid is unstratified the interior velocity is simply the mean of u_T and u_L for then $\lambda = \sigma_V S/2\gamma^2 |\sin\theta|^{\frac{3}{2}} E_V^{\frac{1}{2}}$ is zero. When $\sigma_V S \gg 2\gamma^2 E_V^{\frac{1}{2}} |\sin\theta|^{\frac{3}{2}}$ the interior zonal velocity is no longer independent of z but varies linearly from u_L on $z = 0$ to u_T on $z = 1$. Thus as the fluid becomes more and more substantially stratified the interior velocity itself satisfies the no-slip boundary conditions on $z = 0$, 1 with the Ekman layer velocities simultaneously becoming weaker. The very important feature of the spherical geometry is that the condition (4.11) becomes more readily satisfied as θ goes to zero. Hence, in the mechanically driven solution the effect of the stratification becomes more important as the equator is approached and the Ekman layer corrections are reduced. Thus, although the Ekman layer analysis fails at the equator, it is of no consequence, for the effect of the stratification eliminates the need for the Ekman layer. This is true for the general solution, i.e. for large $\lambda(\theta)$ $(\theta \to 0)$

$$u_I \sim u_L + z(u_T-u_L) - \sum_{n=1}^{\infty}\left\{\frac{\alpha_n}{l_n}\left[\frac{\sinh l_n(z-\frac{1}{2}) - 2(z-\frac{1}{2})\sinh\frac{1}{2}l_n}{\cosh\frac{1}{2}l_n}\right] + \frac{\beta_n}{l_n}\left[\frac{\cosh l_n(z-\frac{1}{2}) - \cosh\frac{1}{2}l_n}{\sinh\frac{1}{2}l_n}\right]\right\}\frac{P_n^{(1)}(0)}{\theta}. \tag{4.12}$$

Thus, in general

$$\left.\begin{aligned} u_I &\to u_L \quad \text{on} \quad z = 0,\\ u_I &\to u_T \quad \text{on} \quad z = 1.\end{aligned}\right\} \tag{4.13}$$

as $\theta \to 0$. The Ekman layer to lowest-order fades away as the equator is approached.

If a stream function for the interior meridional motion is defined such that

$$\psi_I(\theta) = \int_\theta^{\frac{1}{2}\pi} w_I \cos\theta \, d\theta,$$

then as $\theta \to 0$

$$\psi_I \sim -\frac{1}{4}\frac{E_V^{\frac{1}{2}}\gamma^2}{\sigma_V S}\sum_{n=1}^{\infty} \alpha_n \frac{2}{l_n} \tanh \tfrac{1}{2} l_n P_n^{(1)}(0). \tag{4.14}$$

Note that in dimensional units w falls from $O(E_V^{\frac{1}{2}})$ near the pole to $O(E_V/\sigma_V S)$ near the equator. Hence in the mechanically driven solution ($\alpha_m = \beta_m = 0$) there is no flux of fluid across the equator and each hemisphere consists of a separate mechanically driven cell with rising fluid near each pole and sinking motion towards the equator. The poleward and equatorward flux of fluid occurs only in the boundary layer.

In the general case T_I and w_I are regular at the equator but u_I (and hence v_I from (4.1b)) are singular at the equator. The singularity in u_I occurs in the interior of the fluid and not on its boundaries.

For future reference it will be convenient to have explicit expressions for the form of the interior solutions as $E_H/E_V \to 0$. Note that if $\nu_V = \nu_H$, $E_H/E_V = O(\delta^2)$. In this limit

$$\begin{aligned}\partial T_I/\partial\theta = {} & -(u_T - u_L)\sin\theta \\ & + \tfrac{1}{2}(\partial/\partial\theta)(H_T + H_L)(z - \tfrac{1}{2}) + \tfrac{1}{2}(\partial/\partial\theta)(H_T - H_L)((z-\tfrac{1}{2})^2 - \tfrac{1}{12}) \\ & + 2\sum_{n=1}^{\infty} \frac{\alpha_n l_n^2 |\sin\theta|^{\frac{3}{2}} E_V^{\frac{1}{2}}}{n(n+1)} P_n^{(1)}(\sin\theta),\end{aligned} \tag{4.15}$$

while

$$\begin{aligned}u_I = {} & u_L + z(u_T - u_L) \\ & + \frac{1}{6}\frac{\partial}{\partial\theta}(H_T - H_L)\left(\frac{z-\frac{1}{2}}{4} - (z-\tfrac{1}{2})^3\right)\frac{1}{\sin\theta} \\ & - \frac{1}{4}\frac{\partial}{\partial\theta}(H_T + H_L)((z-\tfrac{1}{2})^2 - \tfrac{1}{4})\frac{1}{\sin\theta} \\ & - 2\sum_{n=1}^{\infty}\frac{\alpha_n l_n^2 E_V^{\frac{1}{2}} |\sin\theta|^{\frac{3}{2}}(z-\frac{1}{2})}{n(n+1)\sin\theta}\end{aligned} \tag{4.16}$$

and

$$w_I = -\frac{2\gamma^2 E_V^{\frac{1}{2}}}{\sigma_V S}\frac{1}{\cos\theta}\frac{\partial}{\partial\theta}\left\{\cos\theta\sin\theta(u_T - u_L) + \cos\theta \sum_{n=1}^{\infty}\alpha_n P_n^{(1)}(\sin\theta)\right\}, \tag{4.17}$$

where

$$\alpha_n = \frac{1}{l_n^2}\frac{2n+1}{2}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{H_T - H_L}{2}\cos\theta \, P_n(\sin\theta)\, d\theta. \tag{4.18}$$

Finally, if the upper boundary condition (2.2a) is replaced by a no stress condition, the vertical velocity pumped out of the upper Ekman layer is reduced from $O(E_V^{\frac{1}{2}})$ to $O(E_V)$. The interior temperature field is purely conductive (to $O(\sigma_V S)$) and is not coupled to the motion. The velocity field can then be computed from (4.8) with the condition $u(\theta, 0) = u_L$. As in the case discussed above, u_I and v_I are singular in general at the equator while T_I is regular.

5. The equatorial boundary layer

The singularities in the solutions for u_I and v_I reflect primarily the neglect of the viscously controlled meridional velocity in the continuity equation and the subsequent calculation of w_I, T_I and u_I. To remove the unacceptable singularities of the interior solution a boundary layer centred on $\theta = 0$ is required, and the detailed nature of this layer will depend on the ratio E_H/E_V.

Case 1 $\quad E_H = O(E_V)$

In this case it is found that the equatorial boundary layer has a thickness of $O(\sigma_H S)^{\frac{1}{4}}$ and that within this layer the total dynamic fields can be represented, for $\theta \geqslant 0$:

$$u = (\sigma_H S)^{-\frac{1}{4}}\,\hat{u}(\eta, z), \tag{5.1a}$$

$$v = (E_H/\sigma_H S)\,\hat{v}(\eta, z), \tag{5.1b}$$

$$w = E_H/\sigma_H S)^{\frac{3}{4}}\,\hat{w}(\eta, z), \tag{5.1c}$$

$$T = T_I(0,z) + (\sigma_H S)^{\frac{1}{4}}\,\hat{T}(\eta, z), \tag{5.1d}$$

$$p = p_I(0,z) + (\sigma_H S)^{\frac{1}{4}}\,\hat{p}(\eta, z), \tag{5.1e}$$

where
$$\eta = \theta(\sigma_H S)^{-\frac{1}{4}}.$$

The boundary-layer variables satisfy the following scaled equations:

$$-\eta\hat{v} = \tfrac{1}{2}(\partial^2\hat{u}/\partial\eta^2) - \frac{\delta}{(\sigma_H S)^{\frac{1}{2}}}\,\hat{w}, \tag{5.2a}$$

$$\eta\hat{u} = -\frac{\partial\hat{p}}{\partial\eta} + \frac{E_H^2}{2(\sigma_H S)^{\frac{3}{2}}}\frac{\partial^2\hat{v}}{\partial\eta^2}, \tag{5.2b}$$

$$\frac{\partial\hat{p}}{\partial z} = \hat{T} + \delta/(\sigma_H S)^{\frac{1}{2}}\,\hat{u} + \frac{\delta E_H}{\sigma_H S}\frac{1}{2}\frac{\partial^2\hat{w}}{\partial\eta^2}, \tag{5.2c}$$

$$\hat{w} = \tfrac{1}{2}(\partial^2\hat{T}/\partial\eta^2), \tag{5.2d}$$

$$\frac{\partial\hat{w}}{\partial z} + \frac{\partial\hat{v}}{\partial\eta} = 0. \tag{5.2c}$$

If
$$\sigma_H S \gg (\delta E_H, E_H^{\frac{4}{3}}, \delta^2),$$

then $\hat{u}$ satisfies the relatively simple equation

$$\frac{\partial}{\partial\eta}\left[\frac{1}{\eta}\frac{\partial^2\hat{u}}{\partial\eta^2} + \eta\frac{\partial^2\hat{u}}{\partial z^2}\right] = 0. \tag{5.3}$$

As $\eta \to \infty$ the solution of (5.3) must match with the interior solution, hence

$$\lim_{\eta\to\infty}\eta\frac{\partial^2\hat{u}}{\partial z^2} = \lim_{\theta\to 0}\theta\frac{\partial^2 u_I}{\partial z^2} = -\left.\frac{\partial^2 T_I}{\partial\theta\,\partial z}\right|_{\theta=0}. \tag{5.4}$$

Therefore (5.3) may be integrated once to yield

$$\frac{\partial^2\hat{u}}{\partial\eta^2} + \eta^2\frac{\partial^2\hat{u}}{\partial z^2} = \eta C(z), \tag{5.5}$$

where

$$C(z) = \lim_{\theta\to 0}\theta\frac{\partial^2 u_I}{\partial z^2} = -\sum_{n=1}^{\infty}\left[\frac{\alpha_n l_n\sinh l_n(z-\frac{1}{2})}{\cosh\frac{1}{2}l_n} + \frac{\beta_n l_n\cosh l_n(z-\frac{1}{2})}{\sinh\frac{1}{2}l_n}\right]P_n^1(0).$$

As long as $(\sigma_H S)^{\frac{1}{4}} \gg (\delta^2 E_V)^{\frac{1}{6}}$ boundary conditions for (4.5) on $z = 0, 1$ may be obtained by using the Ekman conditions (3.4) and (3.6) for the boundary layer velocity. This yields the condition

$$\hat{u} = 0 \quad \text{on} \quad z = 0, 1. \tag{5.6}$$

Letting, then,

$$\hat{u} = \sum_{k=1}^{\infty} \hat{u}_k \sin k\pi z,$$

$\hat{u}_k$ may be found in the form,

$$\hat{u}_k = -\frac{C_k}{2}\left[\int_0^\eta d\zeta \eta^{\frac{1}{2}}\zeta^{\frac{3}{2}} I_{\frac{1}{4}}\left(\frac{k\pi\zeta^2}{2}\right) K_{\frac{1}{4}}\left(\frac{k\pi\eta^2}{2}\right) + \int_\eta^\infty d\zeta \eta^{\frac{1}{2}}\zeta^{\frac{3}{2}} I_{\frac{1}{4}}\left(\frac{k\pi\eta^2}{2}\right) K_{\frac{1}{4}}\left(\frac{k\pi\zeta^2}{2}\right)\right] + \alpha_k \eta^{\frac{1}{2}} K_{\frac{1}{4}}\left(\frac{k\pi\eta^2}{2}\right), \tag{5.7}$$

where $I_{\frac{1}{4}}(x)$ and $K_{\frac{1}{4}}(x)$ are the modified Bessel functions of the first and second kind, of order one quarter. In (5.7)

$$C_k = 2\int_0^1 C(z) \sin k\pi z \, dz,$$

while α_k is as yet undetermined.

As η becomes large, it is easy to show through the use of the standard asymptotic forms for $I_{\frac{1}{4}}$ and $K_{\frac{1}{4}}$ that

$$\lim_{\eta\to\infty} \hat{u}_k = -(C_k/\eta\pi^2 k^2), \tag{5.8}$$

which assures that $\hat{u}(\eta, z)$ will match to $u_I(\theta, z)$.

On the other hand, it is of interest to investigate the form of the solution near the equator, i.e. as η becomes small.

Again using the asymptotic forms for $I_{\frac{1}{4}}$ and $K_{\frac{1}{4}}$ for small values of the argument, (5.7) becomes

$$\hat{u}_k \sim -\frac{C_k}{2^{\frac{3}{4}}}\eta\frac{(k\pi)^{\frac{1}{4}}}{(\frac{1}{4})!}\int_0^\infty \zeta^{\frac{3}{2}} K_{\frac{1}{4}}\left(\frac{k\pi\zeta^2}{2}\right) d\zeta + \frac{C_k\eta^3}{6} + \alpha_k\frac{\pi}{\sqrt{2}}\left[\left(\frac{k\pi}{4}\right)^{-\frac{1}{4}}\frac{1}{(-\frac{1}{4})!} - \left(\frac{k\pi}{4}\right)^{\frac{1}{4}}\frac{\eta}{(\frac{1}{4})!} + \left(\frac{k\pi}{4}\right)^{\frac{7}{4}}\frac{\eta^4}{(\frac{3}{4})!} + \dots\right]. \tag{5.9}$$

Now for $\theta \leqslant 0$ the solution for $\hat{u}$ may be similarly found and can be written as

$$\hat{u} = \sum_{k=1}^{\infty} \hat{u}_k(\mu) \sin k\pi z,$$

where $\mu = -\theta(\sigma_H S)^{-\frac{1}{4}}$ and where $\hat{u}_k(\mu)$ is given as

$$\hat{u}_k(\mu) = \frac{C_k}{2}\left[\int_0^\mu d\zeta \mu^{\frac{1}{2}}\zeta^{\frac{3}{2}} I_{\frac{1}{4}}\left(\frac{k\pi\zeta^2}{2}\right) K_{\frac{1}{4}}\left(\frac{k\pi\mu^2}{2}\right) + \int_\eta^\infty d\zeta \mu^{\frac{1}{2}}\zeta^{\frac{3}{2}} I_{\frac{1}{4}}\left(\frac{k\pi\mu^2}{2}\right) K_{\frac{1}{4}}\left(\frac{k\pi\zeta^2}{2}\right)\right] + \beta_k \mu^{\frac{1}{2}} K_{\frac{1}{4}}\left(\frac{k\pi\mu^2}{2}\right) \tag{5.10}$$

and for small μ

$$\hat{u}_k(\mu) \sim \frac{C_k}{2^{\frac{3}{4}}}\mu\frac{(k\pi)^{\frac{1}{4}}}{(\frac{1}{4})!}\int_0^\infty \zeta^{\frac{3}{2}} K_{\frac{1}{4}}\left(\frac{k\pi\zeta^2}{2}\right) d\zeta - \frac{C_k}{6}\mu^3 + \beta_k\frac{\pi}{\sqrt{2}}\left[\left(\frac{k\pi}{4}\right)^{-\frac{1}{4}}\frac{1}{(-\frac{1}{4})!} - \left(\frac{k\pi}{4}\right)^{\frac{1}{4}}\frac{\mu}{(\frac{1}{4})!} + \left(\frac{k\pi}{4}\right)^{\frac{7}{4}}\frac{\mu^4}{(\frac{3}{4})!} + \dots\right]. \tag{5.11}$$

Matching $\hat{u}_k$ and $\partial\hat{u}_k/\partial\theta$ on $\eta = 0$ demands that $\alpha_k = \beta_k = 0$. The solution for $\hat{v}$ may be found from $\hat{u}$, viz.

$$\hat{v} = -\frac{1}{2\eta}\frac{\partial^2\tilde{\mu}}{\partial\eta^2}$$

and on $\eta = 0$
$$\hat{v} = -\frac{1}{2}\sum_{k=1}^{\infty} C_k \sin k\pi z = -\tfrac{1}{2}C(z)$$

and matches with the solution for $\theta \leqslant 0$. It is also easy to show that $\hat{w} = O(\eta)$ as y goes to zero so that even the boundary-layer vertical velocity rather fortuitously matches at the equator. In general, higher-order corrections, or the specifications of different boundary conditions at the equator will require the addition of even thinner non-geostrophic or non-hydrostatic regions not considered here interior to the $(\sigma_H S)^{\frac{1}{4}}$ layer.

The nature of the boundary-layer solution and the equatorial form of the zonal velocity is of interest. If the interior temperature gradient is not zero at the equator the singular part of the interior zonal velocity is odd about the equator. It rises to a rather large value $(\sigma_H S)^{-\frac{1}{4}}$ as the equator is approached and then falls to zero at the equator. The geostrophic balance for the zonal wind is maintained and the singularity at the equator is averted because the fluid adjusts itself in the boundary layer to have a vanishing temperature gradient on $\theta = 0$.

Case 2 $\quad E_H \leqslant E_V$

If $E_H \leqslant E_V$ it is only possible to ignore the vertical derivatives in the viscous and diffusive terms in the boundary layer if

$$\sigma_H S \leqslant (E_H/E_V)^2, \tag{5.12}$$

otherwise the vertical derivatives will in fact swamp the horizontal derivatives, in effect, in the boundary layer. It is of interest to examine the nature of the solution when (5.12) is not satisfied. In this case the Coriolis force due to the poleward velocity must be balanced against the viscous forces due to *vertical* diffusion.

This leads to a boundary layer, surprisingly enough, of a very similar form as in case 1. Its thickness is of order $(\sigma_V S)^{\frac{1}{4}}$ and within this region the dynamic variables can be represented in the form,

$$u = (\sigma_V S)^{-\frac{1}{4}}\,\overline{u}(\eta, z), \tag{5.13a}$$

$$v = \frac{E_V}{(\sigma_V S)^{\frac{1}{2}}}\,\overline{v}(\eta, z), \tag{5.13b}$$

$$w = \frac{E_V}{(\sigma_V S)^{\frac{3}{4}}}\,\overline{w}(\eta, z) + \frac{E_V}{\sigma_V S}\,w_I(0), \tag{5.13c}$$

$$T = T_I(0, z) + (\sigma_V S)^{\frac{1}{4}}\,\overline{T}(\eta, z), \tag{5.13d}$$

$$p = p_I(0, z) + (\sigma_V S)^{\frac{1}{4}}\,\overline{p}(\eta, z), \tag{5.13e}$$

where now $\eta = \theta(\sigma_V S)^{-\frac{1}{4}}$.

The barred variables satisfy

$$-\eta\bar{v} = \frac{1}{2}\frac{\partial^2\bar{u}}{\partial z^2} + \tfrac{1}{2}(\sigma_V S)^{\frac{1}{2}}\frac{E_H}{E_V}\frac{\partial^2\bar{u}}{\partial\eta^2}, \tag{5.14a}$$

$$\eta\bar{u} = -(\partial\bar{p}/\partial\eta), \tag{5.14b}$$

$$\frac{\partial\bar{p}}{\partial z} = \bar{T} + \frac{\delta}{(\sigma_V S)^{\frac{1}{2}}}\bar{u} + \frac{\delta^2 E_H E_V}{2(\sigma_V S)^2}\frac{\partial^2\bar{w}}{\partial\eta^2} + \frac{\delta^2 E_V^2}{2(\sigma_V S)^2}\frac{\partial^2\bar{w}}{\partial z^2}, \tag{5.14c}$$

$$\bar{w} = \frac{1}{2}\frac{\partial^2\bar{T}}{\partial z^2} + \frac{1}{2}\frac{E_H}{E_V}\frac{\sigma_V}{\sigma_H}\frac{1}{(\sigma_V S)^{\frac{1}{2}}}\frac{\partial^2\bar{T}}{\partial\eta^2}, \tag{5.14d}$$

$$\frac{\partial\bar{w}}{\partial z} + \frac{\partial\bar{v}}{\partial\eta} = 0. \tag{5.14e}$$

In the representation for the variables in (4.14) it is important to note that in the limit of vanishing $E_H|E_V$, T_I is a quadratic function of z and w_I is independent of z.

It is convenient to eliminate all variables in favour of $\bar{T}$ to yield

$$\left[\frac{\partial^2}{\partial\eta^2} + \eta^2\frac{\partial^2}{\partial z^2}\right]\frac{\partial^2\bar{T}}{\partial\eta\,\partial z} = 0. \tag{5.15}$$

On $z = 0, 1$ $\partial^2\bar{T}/\partial\eta\,\partial z$ is known, so it is convenient to write

$$\frac{\partial^2\bar{T}}{\partial\eta\,\partial z} = \sum_{k=1}^{\infty} S_k(\eta)\sin k\pi z \tag{5.16}$$

yielding

$$\frac{\partial^2}{\partial\eta^2}\left(\frac{S_k}{\eta^2}\right) - k^2\pi^2\eta^2\left(\frac{S_k}{\eta^2}\right) = 2\pi k\left\{\frac{\partial^2\bar{T}}{\partial z\,\partial\eta}(\eta,1)(-1)^k - \frac{\partial^2\bar{T}(0)}{\partial z\,\partial\eta}\right\}$$
$$= \lim_{\theta\to 0} -2\pi k\left\{\theta\frac{\partial^2\bar{u}}{\partial z^2}(\theta,1)(-1)^k - \theta\frac{\partial^2\bar{u}}{\partial z^2}(\theta,0)\right\}. \tag{5.17}$$

The general solution of (5.17) which is finite as $\eta\to\infty$ can be written in the form

$$\frac{S_k}{\eta^2} = 2\alpha_k\eta^{\frac{1}{2}}K_{\frac{1}{4}}\left(\frac{k\pi\eta^2}{2}\right)$$
$$+\lim_{\theta\to 0} k\pi\left[\theta\frac{\partial^2 u_I}{\partial z^2}(\theta,1)(-1)^k - \theta\frac{\partial^2 u_I}{\partial z^2}(\theta,0)\right]$$
$$\times\left[\int_0^\eta d\zeta\,\eta^{\frac{1}{2}}\zeta^{\frac{1}{2}}I_{\frac{1}{4}}\left(\frac{k\pi\zeta^2}{2}\right)K_{\frac{1}{4}}\left(\frac{k\pi\eta^2}{2}\right) + \int_\eta^\infty d\zeta\,\eta^{\frac{1}{2}}\zeta^{\frac{1}{2}}I_{\frac{1}{4}}\left(\frac{k\pi\eta^2}{2}\right)K_{\frac{1}{4}}\left(\frac{k\pi\zeta^2}{2}\right)\right]. \tag{5.18}$$

Since

$$\bar{v} = -\frac{1}{2\eta}\frac{\partial^2\bar{u}}{\partial z^2} = \frac{1}{2\eta^2}\frac{\partial^2\bar{T}}{\partial\eta\,\partial z} = \sum_{k=1}^{\infty}\bar{v}_k\sin k\pi z, \tag{5.19}$$

we have, using (54.16),

$$\bar{v}_k(\eta) = \alpha_k\eta^{\frac{1}{2}}K_{\frac{1}{4}}\left(\frac{k\pi\eta^2}{2}\right) - \frac{\pi k}{2}\lim_{\theta\to 0}\left[\theta\left(\frac{\partial^2 u_I}{\partial z^2}(\theta,1)(-1)^k - \frac{\partial^2 u_I}{\partial z^2}(\theta,0)\right)\right]$$
$$\times\left[\int_0^\eta d\zeta\,\eta^{\frac{1}{2}}\zeta^{\frac{1}{2}}I_{\frac{1}{4}}\left(\frac{k\pi\zeta^2}{2}\right)K_{\frac{1}{4}}\left(\frac{k\pi\eta^2}{2}\right) + \int_\eta^\infty d\zeta\,\eta^{\frac{1}{2}}\zeta^{\frac{1}{2}}I_{\frac{1}{4}}\left(\frac{k\pi\eta^2}{2}\right)K_{\frac{1}{4}}\left(\frac{k\pi\zeta^2}{2}\right)\right]. \tag{5.20}$$

In the limit of small E_H/E_V, u_I is a cubic function of z and hence $\partial^2 u_I/\partial z^2$ is a linear function of z. Thus, if

$$\frac{\partial^2 u_I}{\partial z^2} = \sum_{k=1}^{\infty}\left(\frac{\partial^2 u_I}{\partial z^2}\right)_k\sin k\pi z$$

it is easy to show that

$$\left(\frac{\partial^2 \bar{u}_I}{\partial z^2}\right)_k = -\frac{2}{k\pi}\left\{\frac{\partial^2 u_I}{\partial z^2}(\theta, 1)(-1)^k - \frac{\partial^2 u_I}{\partial z^2}(\theta, 0)\right\}. \tag{5.21}$$

Therefore,

$$\bar{v}_k = \frac{\pi^4 k^4}{4}\lim_{\theta\to 0}\theta u_{Ik}\left\{\int_0^\eta d\zeta\,\eta^{\frac{1}{2}}\zeta^{\frac{1}{2}} I_{\frac{1}{4}}\left(\frac{k\pi\zeta^2}{2}\right)K_{\frac{1}{4}}\left(\frac{k\pi\eta^2}{2}\right) + \int_\eta^\infty d\zeta\,\eta^{\frac{1}{2}}\zeta^{\frac{1}{2}} I_{\frac{1}{4}}\left(\frac{k\pi\eta^2}{2}\right)K_{\frac{1}{4}}\left(\frac{k\pi\zeta^2}{2}\right)\right\} + \alpha_k \eta^{\frac{1}{2}} K_{\frac{1}{4}}\left(\frac{k\pi\eta^2}{2}\right), \tag{5.22}$$

$$\bar{u}_k = \frac{\pi^2 k^2}{2}\lim_{\theta\to 0}\theta u_{Ik}\left\{\int_0^\eta d\zeta\,\eta^{\frac{1}{2}}\zeta^{\frac{1}{2}} I_{\frac{1}{4}}\left(\frac{k\pi\zeta^2}{2}\right)K_{\frac{1}{4}}\left(\frac{k\pi\eta^2}{2}\right) + \int_\eta^\infty d\zeta\,\eta^{\frac{1}{2}}\zeta^{\frac{1}{2}} I_{\frac{1}{4}}\left(\frac{k\pi\eta^2}{2}\right)K_{\frac{1}{4}}\left(\frac{k\pi\zeta^2}{2}\right)\right\} + \frac{2\alpha_k}{k^2\pi^2}\eta^{\frac{3}{2}} K_{\frac{1}{4}}\left(\frac{k\pi\eta^2}{2}\right), \tag{5.23}$$

where
$$u_{Ik} = 2\int_0^1 u_I(\theta, z)\sin k\pi z\,dz$$
and
$$\bar{u}_k = 2\int_0^1 \bar{u}(\eta, z)\sin k\pi z\,dz.$$

The asymptotic form of $\bar{u}_k$ as η becomes large is easy to work out, and for large η it can be shown that

$$\bar{u}_k \sim (\theta/\eta)\,u_{Ik} = (\sigma_V S)^{\frac{1}{4}} u_{Ik}. \tag{5.24}$$

Since the scaling amplitude of $\bar{u}$ is greater than u_I by $(\sigma_V S)^{-\frac{1}{4}}$, (5.24) shows that the boundary-layer zonal velocity will smoothly match into the interior.

For small η, (5.22) and (5.23) become

$$\bar{u}_k \sim \frac{\pi^2 k^2}{2^{\frac{3}{2}}}\lim_{\theta\to 0}[\theta u_{Ik}]\,\eta^2\int_0^\infty \zeta^{\frac{1}{2}} K_{\frac{1}{4}}\left(\frac{k\pi\zeta^2}{2}\right)d\zeta - \frac{\pi^2 k^2}{2}\lim_{\theta\to 0}[\theta u_{Ik}]\,\eta^3 + \frac{\alpha_k}{\pi k^2}\sqrt{2}\left[\frac{(k\pi)^{-\frac{1}{4}}\eta}{(-\frac{1}{4})!} - \frac{(k\pi)^{\frac{1}{4}}\eta^2}{(\frac{1}{4})!} + O(\eta^5)\right], \tag{5.25}$$

$$\bar{v}_k = \frac{\pi^4 k^4}{4\sqrt{2}}\lim_{\theta\to 0}[\theta u_{Ik}]\,\eta\,\frac{(k\pi)^{\frac{1}{4}}}{(\frac{1}{4})!}\int_0^\infty \zeta^{\frac{1}{2}} K_{\frac{1}{4}}\left(\frac{k\pi\zeta^2}{2}\right)d\zeta - \pi^4 k^4 \lim_{\theta\to 0}[\theta u_{Ik}]\,\eta^2 + \frac{\alpha_k \pi}{\sqrt{2}}\left[\frac{(k\pi)^{-\frac{1}{4}}}{(-\frac{1}{4})!} - \frac{(k\pi)^{\frac{1}{4}}}{(\frac{1}{4})!}\eta + O(\eta^4)\right]. \tag{5.26}$$

Now the appropriate solutions for $\theta \leqslant 0$ are identical to (5.22) and (5.23) with the exception that (θu_{Ik}) is replaced by $-(\theta u_{Ik})$, η is replaced by $\mu = -\theta(\sigma_V S)^{-\frac{1}{4}}$ while α_k is replaced by β_k. Matching the zonal and northward velocities, as well as the zonal shear yields only

$$\alpha_k = \beta_k.$$

However to match the temperature on $\theta = 0$ to order $(\sigma_V S)^{\frac{1}{4}}$ requires that $\partial\bar{w}/\partial z$ and hence $\partial\bar{v}/\partial\theta$ be continuous and this demands that

$$\alpha_k = \beta_k = 0,$$

which completes the solution.

It is quite remarkable that the structure of the equatorial region is so similar in the two cases $E_H = O(E_V)$ and $E_H \ll E_V$. Of course there are important differences. When $E_H \ll E_V$ the meridional velocity v_I is much less singular at the equator than in the case when $E_H = O(E_V)$. This is simply because in the latter case $v_I \sim E_H \partial^2 u_I/\partial\theta^2$ while in the former $v_I \sim E_V \partial^2 u_I/\partial z^2$. This reflected in the differing scaling amplitudes for v in the two cases discussed in this section. Similarly, u vanishes more rapidly at the equator in the second case for essentially the same reason. For since $\eta\bar{v} = \frac{1}{2}\partial^2\bar{u}/\partial z^2$ $\bar{u}/\bar{v}$ is $O(\eta)$ as $\eta \to 0$. Before $\bar{u}$ vanishes at the equator it rises like θ^{-1} to an amplitude $(\sigma_V S)^{-\frac{1}{4}}$ larger than its interior value and only then returns to zero.

The appearance of $S^{\frac{1}{4}}$ (for σ_H or σ_V of $O(1)$) as the natural non-dimensional length scale for the boundary layer can be explained simply as follows.

When the Rossby deformation radius, $L_R = (g\alpha\Delta T_V D)^{\frac{1}{2}}/2\Omega\sin\theta$, is of the same order as the length scale, l, of the motion the constraints of stratification and rotation are equally important. In the boundary layer near the equator the singularity in the geostrophic solution is removed by vorticity diffusion when internal vortex tube stretching, proportional to $\partial w/\partial z$ becomes coupled to the temperature field. This can only occur when $l = O(L_R)$ or when

$$l/R = \sin\theta = (g\alpha\Delta T_V D)^{\frac{1}{4}}/(2\Omega)^{\frac{1}{2}} = S^{\frac{1}{4}}.$$

In general, this length scale, for the coupling of density and vorticity fields, will naturally occur as an intrinsic scale criterion. When $\sin\theta$ is $O(1)$ the critical length scale is $S^{\frac{1}{2}}$ and occurs in the boundary-layer analysis of Barcilon & Pedlosky (1967) and in fact is the natural length scale for baroclinic instability waves for similar coupling reasons (see, for example, Eady 1949). In equatorial regions, as in this paper, the scale is $S^{\frac{1}{4}}$ and this scale also appears as a natural length scale for baroclinic planetary waves centred around the equator (see for example Blandford 1966).

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REFERENCES

BARCILON, V. & PEDLOSKY, J. 1967 A unified linear theory of homogeneous and stratified rotating fluids. *J. Fluid Mech.* **29**, 608–621.

BLANDFORD, R. 1966 Mixed gravity–Rossby waves in the ocean. *Deep-Sea Res.* **13**, 941–961.

CHARNEY, J. G. 1968 The intertropical convergence zone and the Hadley circulation of the atmosphere. (To appear.)

EADY, E. T. 1949 Long waves and cyclone waves. *Tellus*, **1**, 33–52.

PALMEN, E. 1963 General circulation of the tropics. *Proceedings of the Symposium on Tropical Meteorology*, pp. 1–30. New Zealand Meteor. Service, Wellington, New Zealand.

STEWARTSON, K. 1966 On almost rigid rotations. Part 2. *J. Fluid Mech.* **26**, 131–144.